Turbolike Codes on Nonstandard Channel Models

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Is Coding Dead?

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R.I.P.
BLOCK
1948--??

R.I.P.
CONVOLUTIONAL
1955--??

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Special Thanks To

Aamod Khandekar

Ravi Palanki

(Who did most of the work)
**Theorem:** For any (discrete-input memoryless) channel, there exists a number $C$, the channel capacity, such that for any desired data rate $R < C$ and any desired error probability $p > 0$, it is possible to design an encoder-decoder pair that permits the transmission of data over the channel at rate $R$ and decoded error probability $< p$. 
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- $\chi_E(\epsilon, p) =$ the minimum possible *encoding* complexity, in operations *per information bit*. 
How Hard is it to Approach Channel Capacity?

- Desired transmission rate $R = C(1 - \epsilon)$.
- Desired decoder error probability $= p$.
- $\chi_E(\epsilon, p) = \text{the minimum possible encoding complexity, in operations per information bit.}$
- $\chi_D(\epsilon, p) = \text{the minimum possible decoding complexity, in operations per information bit.}$
How Hard is it to Approach Channel Capacity?

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- Desired decoder error probability $= p$.
- $\chi_E(\epsilon, p)$ = the minimum possible *encoding* complexity, in operations *per information bit*.
- $\chi_D(\epsilon, p)$ = the minimum possible *decoding* complexity, in operations *per information bit*.

*For fixed $p$, how do $\chi_E(\epsilon, p)$ and $\chi_D(\epsilon, p)$, behave, as $\epsilon \rightarrow 0$?*
The Classical Results.

**Theorem:** On a discrete memoryless channel of capacity $C$, for any fixed $p > 0$, as $\epsilon \to 0$,

$$\chi_E(\epsilon, p) = O\left(\frac{1}{\epsilon^2}\right)$$

$$\chi_D(\epsilon, p) = 2^{O\left(\frac{1}{\epsilon^2}\right)}.$$
The Classical Results.

**Theorem:** On a discrete memoryless channel of capacity $C$, for any fixed $p > 0$, as $\epsilon \to 0$,

$$
\chi_E(\epsilon, p) = O(1/\epsilon^2)
$$

$$
\chi_D(\epsilon, p) = 2^{O(1/\epsilon^2)}.
$$

**Proof:** Use linear codes with (per-bit) encoding complexity $O(n)$, and ML decoding with complexity $2^{O(n)}$. And $n = O(1/\epsilon^2)$, because of the random coding exponent:

$$
\bar{p} \leq e^{-nE_r(R)}
$$

where

$$
E_r(C(1 - \epsilon)) \approx K\epsilon^2 \quad \text{as} \quad \epsilon \to 0.
$$
The Shannon Challenge
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• For mathematicians: Reduce the decoding complexity to

\[ \chi_D(\epsilon, p) = O\left(\frac{1}{\epsilon}^m\right). \]
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- For mathematicians: Reduce the decoding complexity to
  \[ \chi_D(\epsilon, p) = O\left(\frac{1}{\epsilon^m}\right). \]

- For engineers: Approach the Shannon limit practically!
Has Shannon’s Challenge Already Been Met by Turbo-Codes?
We Should Include Variations
On the Turbo-Theme ("Turbolike" Codes)
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:}


Turbolike Codes Have Certainly Met the Challenge on the Binary Erasure Channel!

\[ \begin{array}{c}
0 & \overset{1-p}{\rightarrow} & 0 \\
p & \downarrow & ? \\
p & \downarrow & ? \\
1 & \overset{1-p}{\rightarrow} & 1 
\end{array} \]

**Theorem:** For the binary erasure channel, for the ensemble of (degree-profile optimized) irregular LDPC codes with iterative belief propagation decoding, as \( \epsilon \to 0 \),

\[ \chi_D(\epsilon, p) = O(\log \frac{1}{\epsilon}) \]

Irregular LDPC Codes, Density Evolution
(Luby, Mitzenmacher, Shokrollahi, Spielman, Stemmann)
Turbolike Codes Appear to Have Met the Challenge on the Additive Gaussian Noise Channel (I)

Irregular LDPC Codes, Density Evolution
(Chung, Forney, Richardson, Urbanke, 2001)
Turbolike Codes Appear to Have Met the Challenge on the Additive Gaussian Noise Channel (II)

Turbo-Hadamard Codes (Ping, Leung, Wu, ISIT 2001)
A Conjecture

Turbolike codes meet the Shannon challenge for any symmetric binary input channel. To be precise: there exists a sequence of turbolike code ensembles plus matched iterative belief propagation decoding algorithms, such that for any fixed $p$, as $\epsilon \to 0$,

$$\chi_E(\epsilon, p) = O\left(\log \frac{1}{\epsilon}\right)$$

$$\chi_D(\epsilon, p) = O\left(\frac{1}{\epsilon} \log \frac{1}{\epsilon}\right)$$

(Khandekar and McEliece, ISIT 2001)
Three Garden-Variety SBIC’s

\[ z = N(0, \sigma^2) \]

Binary Symmetric

Binary Erasure

Binary Input, Additive Gaussian Noise
The Generalization (Gallager, 1963)

Definition: A symmetric binary-input channel is a memoryless, discrete-time channel with
• Input alphabet $X = \{+1, -1\}$.
• Output alphabet $Y \subseteq \text{Real Numbers}$.
• Transition probabilities

$$p(y|x = +1) = f(y)$$
$$p(y|x = -1) = f(-y).$$
Examples of SBIC’s

• The Binary Erasure Channel:

\[ f(y) = (1 - p)\delta(y - 1) + p\delta(y). \]

• The Binary Symmetric Channel:

\[ f(y) = (1 - p)\delta(y - 1) + p\delta(y + 1). \]

• Additive Gaussian Noise:

\[ f(y) = K \exp\left(\frac{(y - 1)^2}{2\sigma^2}\right). \]

• Fast Rayleigh Fading (noncoherent model):

\[ f(y) = \begin{cases} 
K \exp\left(-\frac{y}{A}\right) & \text{if } y \geq 0 \\
K \exp\left(y\left(1 + \frac{A}{y}\right)\right) & \text{if } y < 0.
\end{cases} \]
But What About Non-SBIC’s, i.e., (Memoryless) Channels that are
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But What About Non-SBIC’s, i.e.,
(Memoryless) Channels that are

\[
\begin{align*}
0 & \rightarrow 1 & \rightarrow 0 \\
1 & \rightarrow 1-p & \rightarrow 1
\end{align*}
\]

Nonsymmetric?

Nonbinary?
But What About Non-SBIC’s, i.e., (Memoryless) Channels that are

Nonsymmetric?

Nonbinary?

Multiuser?
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Etc.?
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Etc.?
The Simplest Nonsymmetric Channel: The $Z$-Channel
The Simplest Nonsymmetric Channel: The Z-Channel

\[ \begin{align*}
0 & \quad 1 & \quad 0 \\
1 & \quad 1-p & \quad 1
\end{align*} \]

Capacity

\[ \begin{align*}
p \quad & \quad 0 & \quad 0.2 & \quad 0.4 & \quad 0.6 & \quad 0.8 & \quad 1 \\
\text{Capacity} & \quad 1 & \quad 0.8 & \quad 0.6 & \quad 0.4 & \quad 0.2 & \quad 0
\end{align*} \]
An Experiment on the Z-channel
(Rate $\frac{1}{3}$ Repeat-Accumulate Code, $k = 1000$.)

Performance of rate 1/3 RA codes, $k=1000$, #iterations=50, on a Z-channel

$P^* p_0$

$-\log_{10} p$
Can We Deal More Honestly With the Problem?
**Theorem.** Binary linear codes can be used to achieve capacity on an arbitrary discrete memoryless channel.

**Proof:** Encoding algorithm:
An Old Theorem of Gallager (1968)

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**Proof:** Encoding algorithm:

```
<table>
<thead>
<tr>
<th>Binary Encoder</th>
<th>Mapper</th>
<th>3-input DMC</th>
</tr>
</thead>
<tbody>
<tr>
<td>000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>001</td>
<td></td>
<td>a</td>
</tr>
<tr>
<td>010</td>
<td></td>
<td>b</td>
</tr>
<tr>
<td>110</td>
<td></td>
<td>c</td>
</tr>
<tr>
<td>100</td>
<td></td>
<td></td>
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<tr>
<td>101</td>
<td></td>
<td></td>
</tr>
<tr>
<td>111</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```

- 000 $\rightarrow$ a  ($p = 3/8$)
- 001 $\rightarrow$ a  ($p = 3/8$)
- 010 $\rightarrow$ a  ($p = 3/8$)
- 100 $\rightarrow$ b  ($p = 3/8$)
- 101 $\rightarrow$ b  ($p = 3/8$)
- 110 $\rightarrow$ c  ($p = 1/4$)
- 111 $\rightarrow$ c  ($p = 1/4$)
An Old Theorem of Gallager (1968)

**Theorem.** *Binary linear codes can be used to achieve capacity on an arbitrary discrete memoryless channel.*

**Proof:** Encoding algorithm:
“Unfortunately, the problem of finding decoding algorithms is not so simple.” —R.G.G.
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Joint Decoding-Demapping

How does the Demapper interact with the Iterative Decoder?
A Bayesian Network For a 4:1 Mapper
Problem: infer $z_1, z_2, z_3, z_4$ after observing $y$.

Mapping:
$$x = f(z_1, z_2, z_3, z_4)$$

Noisy channel:
$$q(y|x)$$
The Corresponding Junction Tree

(Iterative Decoder)

\[ x = f(z_1, z_2, z_3, z_4) \]

\[ q(y^e | x) \]
Example: The 16-ary Symmetric Channel

\[ \begin{array}{c}
0000 \\
0001 \\
0010 \\
0011 \\
0100 \\
0101 \\
0110 \\
0111 \\
1000 \\
1001 \\
1010 \\
1011 \\
1100 \\
1101 \\
1110 \\
1111
\end{array} \]

\[ (1-p) \]

\[ \frac{p}{15} \text{ each} \]
An Experiment on the 16-ary Symmetric Channel

\( R = \frac{1}{3} \) RA Code, \( k = 1000 \)
This Approach has Proved Effective on the 2D Additive White Gaussian Channel

(graph due to Divsalar and Pollara)
A Simple Multiuser (Multiaccess) Channel

(w and x must transmit independently to y.)
A Tanner Graph for a \((2, 3)\) LDPC Code
Splitting the Graph (I)
Splitting the Graph (II)
A Tanner Graph for a Multiaccess LDPC Code
\((n = 6, R_1 = 5/6, R_2 = 1/2)\).

\[y_i = \text{channel response to } (w_i, x_i).\]
The Corresponding Junction Graph

\[ \chi(\mathbf{w}, \mathbf{x}) = \begin{cases} 
1 & \text{if even parity} \\
0 & \text{if odd parity.} 
\end{cases} \]
Example: The Binary Adder Channel

\[
\begin{array}{ccc}
  x_1 & + & x_2 \\
  \downarrow &  & \downarrow \\
  x & & y \\
\end{array}
\]

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>
BAC Joint Decoding Example
BAC Joint Decoding Example
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? 1 ?

+ 0 +

1

+ 1 +

2

2
BAC Joint Decoding Example
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The diagram illustrates a joint decoding example, with each node representing a time step or frame. The numbers indicate the state of the decoder at each step. The connections between the nodes show the transitions and decisions made in the decoding process.
BAC Joint Decoding Example
BAC Joint Decoding Example
The Capacity Region for the BAC

$R_1 + R_2 = 1.50$
Experimental Results Based on Splitting Irregular RA Codes ($n = 10000$)
A Theorem.

Theorem.  *Turbolike codes meet Shannon’s Challenge on the BAC (without the need to timeshare).*

Proof. Use density evolution. It’s almost exactly like the BEC.

(Palanki, Khandekar and McEliece, Allerton 2001)
Conclusions?
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• Although *applications* of turbolike codes to nonstandard channels are just beginning to appear, *graph-based iterative message-passing* may be a panacea.
Conclusions?

- On *standard channel models* (SBIC’s), coding technology is quite mature.
- Still, *theory* has a lot of catching up to do. Density evolution may not be enough.
- Although *applications* of turbolike codes to nonstandard channels are just beginning to appear, *graph-based iterative message-passing* may be a panacea.

- Coding isn’t dead quite yet!